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Two-level system coupled to a boson mode: the large *n* limit

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Abstract. The classical limit $n \gg 0$ is investigated for a two-level system coupled to a boson mode. The usual formulations are found to neglect a shift λ^2 of the spectrum where λ is the strength of the coupling. A scaling law for the highly excited energy levels is derived and tested numerically. An insight is obtained into the characteristic properties of the energy level distribution which is of interest in connection with possible 'quantum chaotic' behaviour. The interpretation of the asymptotically effective Hamiltonian as a one-dimensional tight-binding model for the two-band Stark ladder is discussed.

1. Introduction

The Hamiltonian of a two-level system coupled to a boson mode is

$$H = N + \lambda(a + a^{+})\sigma_x + \tfrac{1}{2}\Delta\sigma_z. \tag{1.1}$$

Here $N = a^{+}a$ and a, a^{+} are the usual annihilation and creation operators for bosons. The Pauli matrices σ_i describe the two-level system with the levels separated by Δ and λ is the coupling constant. This Hamiltonian serves as a non-trivial model in spin resonance, quantum optics and for various problems in solid state physics (for a long list of references we refer to those cited by Reik *et al* (1982) and Graham and Höhnerbach (1984a)). Since it was found that its semiclassical counterpart shows chaotic motion (Zaslavsky 1981, Milonni *et al* 1983) the system (1.1) now also serves as a model for the discussion of 'quantum chaos' (Graham and Höhnerbach 1984a, b, c, Kuś 1985a, Steeb *et al* 1985). Because one of the criteria proposed for the discrimination between chaotic and non-chaotic quantum systems is based on statistical properties of the energy levels (Percival 1973, Berry and Tabor 1977, Pechukas 1983) the general structure of the spectrum in respect of the parameters λ and Δ has attracted renewed interest. With the noteworthy exception of a set of exact isolated solutions discovered recently by Reik *et al* (1982) (see also Kuś 1985b, Kuś and Lewenstein 1986) the spectrum is not known in explicit analytic form. Instead, there a large number of approximations have been studied, valid in different regimes (for a review see Graham and Höhnerbach (1984a)). In addition, the spectrum has been investigated extensively by numerical methods (Reik *et al* 1982, Graham and Höhnerbach 1984a, Kuś 1985a, Steeb *et al* 1985).

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In this paper we concentrate on the limit of large boson numbers $n = \langle N \rangle \gg 0$ which we call the large n limit. Because for given values of λ and Δ all levels beyond a limited number of n fall into this regime we may expect to get an insight into various characteristic properties of the numerically calculated level distributions (Kuś 1985a, Steeb *et al* 1985) from the large n limit.

This limit has been formulated in two versions which look very different at first sight. In one of these formulations the highly excited boson mode is treated as an oscillating classical field. By the use of Floquet's theorem the explicitly time-dependent spin problem may be transformed into an eigenvalue problem. The properties of the corresponding Floquet Hamiltonian have been studied in great detail numerically and in various analytic approximations by Autler and Townes (1955). The second version of the large n limit is obtained directly from the Hamiltonian (1.1) when the n-dependent off-diagonal matrix elements resulting from the coupling term are replaced by the constant elements $\lambda n_0^{1/2}$. The equivalence of both versions was first pointed out by Shirley (1965).

When applying the second formulation to the simple case $\Delta = 0$ we find that the resulting spectrum differs from the exact one by an n-independent shift given by λ^2 which contradicts the claimed convergence of both spectra for $n \to \infty$ (§ 2). We take this discrepancy as a reason to reconsider this limit.

A more careful investigation indeed shows that the usual expressions for $n \gg 0$ have to be corrected by a shift λ^2 (§ 3). The reason why this shift has escaped attention is probably due to the fact that it does not show up in time-dependent problems (e.g. transition probabilities). It is essential, however, when we are interested in the absolute energy levels. This finding shows that the large n limit is not entirely trivial, which we find to be caused by the approximation of the unbounded operators a and a^+ by bounded ones.

In § 4 we discuss the properties of the proper effective Hamiltonian for the highly excited states. This Hamiltonian has a number of simplifying symmetries giving rise to a ladder structure of the spectrum. As a consequence it follows that in the classical limit the spectrum of the two-level system obeys a scaling law (§ 5). For given Δ the highly excited levels even for varying λ are all mapped by this law onto a single universal curve. Numerical tests show that the agreement with the asymptotic behaviour is remarkably good, even for small values of n. In combination with the known properties of the strong coupling limit we obtain a rather complete understanding of the level distribution, thus giving an explanation to the numerical results of Kuś (1985a) and Steeb *et al* (1985).

In § 6 we point out that in the large n limit the effective Hamiltonian has an interesting interpretation as a one-dimensional tight-binding model for the two-band Stark ladder. In the basis of the Bloch wavefunctions the eigenvalue problem turns out to be formally equivalent to the behaviour of a spin in an oscillating classical field. This formulation coincides with the first version of the large n limit, thus illuminating it from a different point of view.

2. The large *n* limit

The discussion of the Hamiltonian (1.1) is facilitated by performing first a unitary transformation which diagonalises H with respect to the spin variables (see Shore and

Sander 1973). The result is

$$H = N + \lambda(a + a^+) + \tfrac{1}{2}\Delta R \sigma_z \tag{2.1}$$

where the operator

$$R = (-1)^N \tag{2.2}$$

obeys

$$R = R^+ = R^{-1} \tag{2.3}$$

$$[R, N]_- = 0 \qquad [R, a^{(+)}]_+ = 0. \tag{2.4}$$

Hence, it suffices to consider the purely bosonic Hamiltonian

$$H = N + \lambda(a + a^+) + \tfrac{1}{2}\Delta(-1)^N. \tag{2.5}$$

The spectrum of the original Hamiltonian (1.1) or (2.1) is given by the superposition of the spectra of (2.5) for the values $\Delta = \pm|\Delta|$. The unitary transformation of H (2.5) by means of R has the simple effect $\lambda \to -\lambda$. Hence, with respect to the coupling we may restrict ourselves to the range $\lambda \geqslant 0$. For later comparison we recall

$$[a, N]_- = a \qquad [N, a^+]_- = a^+ \tag{2.6}$$

$$[a, a^+]_- = 1. \tag{2.7}$$

In order to make the N dependence of the coupling term more explicit we define the operator

$$T = \sum_0^\infty |n\rangle\langle n+1|. \tag{2.8}$$

In the basis of the occupation number states $\{|n\rangle;\, n = 0, 1, 2, \ldots\}$ T has constant off-diagonal matrix elements and obeys

$$[T, N]_- = T \qquad [N, T^+]_- = T^+ \tag{2.9}$$

$$TT^+ = 1 \qquad T^+T = 1 - |0\rangle\langle 0| \qquad [T, T^+]_- = |0\rangle\langle 0| \tag{2.10}$$

$$T^+|n\rangle = |n+1\rangle \qquad T|n\rangle = \begin{cases} 0 & n = 0 \\ |n-1\rangle & n > 0. \end{cases} \tag{2.11}$$

Then

$$a = TN^{1/2} \qquad a^+ = N^{1/2}T^+ \tag{2.12}$$

and the Hamiltonian (2.5) may be written as

$$H = N + \lambda(TN^{1/2} + N^{1/2}T^+) + \tfrac{1}{2}\Delta(-1)^N. \tag{2.13}$$

At this point we want to make a remark on the numbering of the energy levels. From numerical calculations it is found that as a function of λ the levels avoid crossings. Hence, for given λ and Δ we may follow a particular level in a unique way backwards to the simple case $\lambda = 0$ where

$$E(n, \lambda = 0, \Delta) = n + \tfrac{1}{2}\Delta(-1)^n \qquad n = 0, 1, 2, \ldots \tag{2.14}$$

with n being the eigenvalue of N. According to this rule we may use n as a quantum number for arbitrary values of λ and Δ although N does not commute with H when $\lambda \neq 0$. This convention for the denumeration of the levels is the most convenient one for the general discussion of the large n limit. It is to be noted, however, that according to this rule $E(n, \lambda, \Delta)$ is no more a strictly monotone function of n for $|\Delta|>1$ because then for $\lambda=0$ the two ladders formed by the levels (2.14) with even and odd n respectively have crossed each other at least once. This point has to be kept in mind when the nearest-neighbour spacing distribution function is discussed in the large n limit. For this reason we will switch later on to an equivalent but strictly monotone numbering of the levels.

The usual formulation of the large n limit is based on the observation that in the vicinity of some large boson number n_0 the variation of the off-diagonal matrix elements of H is only of order $\lambda/n_0^{1/2}$. From this behaviour it is suggested that the n_0th level is well approximated for $n_0 \gg 0$, $n_0 \gg |\Delta|$, λ^2 by the corresponding level of the Hamiltonian with constant off-diagonal matrix elements

$$H_{n_0} = N + \lambda n_0^{1/2}(T+T^+) + \tfrac{1}{2}\Delta(-1)^N. \tag{2.15}$$

It is assumed that in the limit $n_0 \to \infty$ the agreement becomes exact:

$$\lim_{n_0\to\infty} [E(n_0, \lambda, \Delta) - E_{n_0}(n_0, \lambda, \Delta)] = 0. \tag{2.16}$$

As was observed by Graham and Höhnerbach (1984a) hardly any simplification is achieved by the approximation (2.15), neither with respect to the analytical structure nor for numerical calculations. To a large extent the difficulties stem from the boundary effects which mainly influence the low energy levels. Being interested in the limit $n_0 \to \infty$ we can avoid these effects by allowing n to take on negative values as well. Explicitly, consider the Hilbert space spanned by the basis $\{|n\rangle\}$ where n runs from minus infinity to infinity. Let

$$\bar{N} = \sum_{-\infty}^{+\infty} n|n\rangle\langle n| \qquad \bar{T} = \sum_{-\infty}^{+\infty} |n\rangle\langle n+1| \tag{2.17}$$

then the spectrum of the approximate Hamiltonian

$$\bar{H}_{n_0} = \bar{N} + \lambda n_0^{1/2}(\bar{T}+\bar{T}^+) + \tfrac{1}{2}\Delta(-1)^{\bar{N}} \tag{2.18}$$

becomes identical with that of H_{n_0} in the limit $E \to +\infty$ and the relation (2.16) is equivalent to

$$\lim_{n_0\to\infty} [E(n_0, \lambda, \Delta) - \bar{E}_{n_0}(n_0, \lambda, \Delta)] = 0. \tag{2.19}$$

The great simplification achieved by this formulation is a consequence of the much simpler algebraic properties of the operators $\bar{N}$, $\bar{T}$ and $\bar{T}^+$ when compared with those of N, T and T^+ or N, a and a^+. Whereas the analogue to (2.9) remains unchanged

$$[\bar{T}, \bar{N}]_- = \bar{T} \qquad [\bar{N}, \bar{T}^+]_- = \bar{T}^+ \tag{2.20}$$

we now have, instead of (2.10) and (2.11),

$$\bar{T}\bar{T}^+ = \bar{T}^+\bar{T} = 1 \qquad [\bar{T}, \bar{T}^+]_- = 0 \tag{2.21}$$

$$\bar{T}^+|n\rangle = |n+1\rangle \qquad \bar{T}|n\rangle = |n-1\rangle. \tag{2.22}$$

We see from (2.21) that $\bar{T}$ is a unitary operator whereas T is not. Most important, $\bar{T}$ and $\bar{T}^+$ commute and may be treated like C numbers in a function $F(\bar{T}, \bar{T}^+)$. Retaining all information on the large n limit we consider for these reasons mainly this latter formulation in the following.

When compared with other approximations a peculiarity of the one under consideration is that with each energy level there is associated an individual effective Hamiltonian. These differ merely in different values of the effective coupling $\Lambda = \lambda n_0^{1/2}$. From each of these Hamiltonians we then have to pick a particular level as the best approximation.

We now turn to the question whether the large n limit defined in this way really obeys the relation (2.19) as is claimed. In order to get a first impression we consider the special case $\Delta = 0$ which can be discussed in detail. Here, the original Hamiltonian (2.5) reduces to a displaced harmonic oscillator and

$$E(n, \lambda, \Delta = 0) = n - \lambda^2 \qquad n = 0, 1, 2, \ldots. \tag{2.23}$$

As will be shown in § 3, the Hamiltonian $\bar{H}_{n_0}$ may easily be diagonalised for $\Delta = 0$. We find

$$E_{n_0}(n, \lambda, \Delta = 0) = n \qquad n = 0, \pm 1, \pm 2, \ldots. \tag{2.24}$$

Hence, in the common range $n \geq 0$ the spectra differ by a n-independent shift λ^2 which is in contradiction with the supposed asymptotic behaviour (2.19). From this result obtained for $\Delta = 0$ we draw the general conclusion that $\bar{H}_{n_0}$ does not describe the large n limit properly. We note that the corresponding discussion for H_{n_0} leads to the same conclusion which shows explicitly that in the limit $E \to \infty$ it does not matter whether we work in the extended Hilbert space or not.

3. Renormalised effective Hamiltonian

In order to find out the reason for this failure we return to the decisive step from H to H_{n_0}. By construction, the approximation consists of the replacement of the unbounded operators $TN^{1/2}$ and $N^{1/2}T^+$ in the coupling term by the bounded operators $n_0^{1/2}T$ and $n_0^{1/2}T^+$. Hence, H differs from H_{n_0} by an unbounded operator which represents a rather drastic perturbation. As is known from other examples (anharmonic oscillator, Stark effect) unbounded perturbations have to be treated with some care.

We avoid in our case a direct discussion of this problem by transforming H first by means of the displacement operator

$$D(\lambda) = \exp[\lambda(a^+ - a)] \qquad DD^+ = 1. \tag{3.1}$$

We obtain

$$H = N - \lambda^2 + H_1 \tag{3.2}$$

$$\begin{aligned} H_1 &= \tfrac{1}{2}\Delta D(\lambda)(-1)^N D(\lambda)^+ \\ &= \tfrac{1}{2}\Delta D(2\lambda)(-1)^N \end{aligned} \tag{3.3}$$

where D, $(-1)^N$ and hence H_1 are bounded operators.

We now perform the large n limit in the matrix elements of $D(\lambda)$. Consider

$$\langle n_0 + k | D(\lambda) | n_0 + l \rangle = [(n_0 + l)!/(n_0 + k)!]^{1/2} \lambda^{k-l} \, e^{-1/2\lambda^2} L_{n_0+l}^{(k-l)}(\lambda^2) \tag{3.4}$$

where the $L_m^{(n)}$ are the Laguerre polynomials. Using the asymptotic properties of these polynomials we find in the limit $n_0 \gg 0$, $n_0 \gg \lambda^2$ the approximate expression

$$\langle n_0 + k | D(\lambda) | n_0 + l \rangle \approx J_{k-l}(2\Lambda) \tag{3.5}$$

where

$$\Lambda = \lambda(n_0 + 1/2)^{1/2} \tag{3.6}$$

and where the J_ν are the Bessel functions. By means of the RHS of (3.5) we may define an operator which incorporates all the approximations of the large n limit. We immediately switch to the extended Hilbert space by allowing k and l to take on negative values as well. We thus define

$$\bar{D}(\Lambda)=\sum_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty} J_{k-l}(2\Lambda)|n_0+k\rangle\langle n_0+l|$$
$$=\sum_{-\infty}^{+\infty} J_k(2\Lambda)\bar{T}^{+k}. \tag{3.7}$$

By means of the algebraic relations (2.21) one easily verifies

$$\bar{D}(\Lambda)=\exp[\Lambda(\bar{T}^{+}-\bar{T})]. \tag{3.8}$$

With respect to the algebra generated by $\bar{N}$, $\bar{T}$ and $\bar{T}^{+}$ the operator $\bar{D}(\Lambda)$ plays a role very similar to that of $D(\lambda)$ for N, a and a^{+}. In fact, $\bar{D}(\Lambda)$ is unitary and from (2.20) it follows

$$\bar{D}(\Lambda)\bar{N}\bar{D}(\Lambda)^{+}=\bar{N}-\Lambda(\bar{T}+\bar{T}^{+}). \tag{3.9}$$

On the other hand there is a marked difference in that

$$[\bar{T}^{(+)},\bar{D}]_{-}=0. \tag{3.10}$$

Keeping in mind that Λ depends on n_0 we arrive at the effective Hamiltonian

$$\bar{H}_{n_0}=\bar{N}-\lambda^2+\tfrac{1}{2}\Delta\bar{D}(\Lambda)(-1)^{\bar{N}}\bar{D}(\Lambda)^{+} \tag{3.11}$$

which is unitary equivalent to

$$\bar{H}_{n_0}=\bar{N}+\Lambda(\bar{T}+\bar{T}^{+})+\tfrac{1}{2}\Delta(-1)^{\bar{N}}-\lambda^2. \tag{3.12}$$

For $n_0\gg 0$ we have $\Lambda\approx\lambda n_0^{1/2}$ and a comparison with (2.18) shows that the approximate Hamiltonian (3.12) differs from the usual formulation (2.18) exactly by the shift λ^2 which was found to be missing in (2.24). Combining this result with numerical tests for $\Delta\neq 0$ presented in § 4 we conclude that the renormalised Hamiltonian (3.12) is the correct formulation of the large n limit. In passing we note that $\bar{H}_{n_0}$ reduces for $\Delta=0$ to a diagonal operator when (3.11) is used.

Our discussion can be summarised as follows. Whereas the substitution $a^{(+)}\to n_0^{1/2}\bar{T}^{(+)}$ done directly in H neglects a shift λ^2, the very same substitution done in the exponential (compare (3.1) with (3.8)) gives the proper large n limit. We remark that this rule has to be used with some care because $\bar{T}$ and $\bar{T}^{+}$ commute whereas a and a^{+} do not. For instance, we have the identities

$$D(\lambda)=\exp(-1/2\lambda^2)\exp(\lambda a^{+})\exp(-\lambda a)$$
$$=\exp(1/2\lambda^2)\exp(-\lambda a)\exp(\lambda a^{+}). \tag{3.13}$$

Here the substitution $a^{(+)}\to n_0^{1/2}\bar{T}^{(+)}$ gives different operators which also differ from $\bar{D}(\Lambda)$. The transition from $D(\lambda)$ to $\bar{D}(\Lambda)$ (3.8) is the only one which conserves unitarity.

4. Spectrum of the effective Hamiltonian

Further discussion is simplified by removing the term $-\lambda^2$ from $\bar{H}$. For this purpose we shift all levels by the same amount in the opposite direction. Thus, we henceforth consider

$$H=N+\lambda(a+a^{+})+\lambda^2+\tfrac{1}{2}\Delta(-1)^{N} \tag{4.1}$$

together with

$$\bar{H}_{n_0}=\bar{N}+\Lambda(\bar{T}+\bar{T}^{+})+\tfrac{1}{2}\Delta(-1)^{\bar{N}} \tag{4.2}$$

where $\Lambda=\Lambda(\lambda, n_0)$ is given by (3.6). Now the energy levels for $\Delta=0$ which form the so-called baselines are independent of λ or Λ and are equal to the integers.

Before we go on to discuss H in terms of the sequence of the effective Hamiltonians $\bar{H}_{n_0}$ we ignore for the moment the meaning of Λ. We thus drop the index n_0 and treat Λ and Δ as independent parameters. As was observed by Autler and Townes (1955) for the case of continued fraction expressions for the eigensolutions the Hamiltonian $\bar{H}$ has a number of symmetry properties which imply that the spectrum consists of two ladders with equal spacing which are shifted against each other with respect to Λ and Δ. In algebraic terms we have

$$\bar{T}^2\bar{H}\bar{T}^{+2} = \bar{H}+2. \tag{4.3}$$

Next we define the unitary operator $\hat{U}$ by

$$\hat{U}|n\rangle = |-n\rangle. \tag{4.4}$$

Then

$$\tilde{U}\bar{H}\tilde{U}^{+} = -\bar{H}+1 \tag{4.5}$$

where

$$\tilde{U} = \hat{U}(-1)^{\bar{N}}\bar{T}. \tag{4.6}$$

From (4.3) and (4.5) it follows that for any level $\bar{E}$ there exist levels $\bar{E}+2m$ and $-\bar{E}+2m+1$ with $m=0, \pm 1, \pm 2, \dots$ which form the two ladders mentioned above. The whole spectrum may be parametrised as

$$\bar{E}(n, \Lambda, \Delta) = n+\tfrac{1}{2}\bar{\Delta}(-1)^n \tag{4.7}$$

$$\bar{\Delta} = \bar{\Delta}(\Lambda, \Delta). \tag{4.8}$$

We thus have obtained from the symmetries (4.3) and (4.5) the n dependence of the spectrum in explicit form. This result forms the basis for our further discussion of the large n limit of the two-level system.

Because of the ladder structure the parametrisation (4.7) is not unique. Different choices merely mean that we label the levels in different ways. As in the case of H it is sufficient to choose a labelling for $\Lambda = 0$. The numbering in terms of the eigenvalues of $\bar{N}$ which we used in the previous sections is obtained by the condition

$$\bar{\Delta}(\Lambda=0, \Delta) = \Delta. \tag{4.9}$$

Here $\bar{E}(n, \Lambda, \Delta)$ is a strictly monotone function of n only for $|\Delta| < 1$. As an alternative we now introduce the monotonic numbering of the levels for all values of Δ. For this purpose we decompose $\frac{1}{2}\Delta$ into an integer part n_Δ and the rest $\frac{1}{2}\Delta_0$

$$\tfrac{1}{2}\Delta = n_\Delta + \tfrac{1}{2}\Delta_0 \qquad |\Delta_0| \leq 1. \tag{4.10}$$

Correspondingly we write

$$\tfrac{1}{2}\bar{\Delta}(\Lambda, \Delta) = n_\Delta + \tfrac{1}{2}\bar{\Delta}_0(\Lambda, \Delta) \tag{4.11}$$

$$\bar{\Delta}_0(\Lambda=0, \Delta) = \Delta_0. \tag{4.12}$$

The spectrum is then given as

$$\bar{E}(\nu, \Lambda, \Delta) = \nu + \tfrac{1}{2}\bar{\Delta}_0(\Lambda, \Delta)(-1)^{\nu+n_\Delta} \tag{4.13}$$

where we here use the greek letter $\nu = 0, \pm 1, \pm 2, \dots$ in order to distinguish this labelling from the one by the n. Explicitly, the relationship between the two versions is given by

$$\nu = \nu(n) = n + n_\Delta(-1)^n \tag{4.14}$$

$$n = n(\nu) = \nu - n_\Delta(-1)^{\nu+n_\Delta}. \tag{4.15}$$

By construction, both numberings coincide for $n_\Delta = 0$.

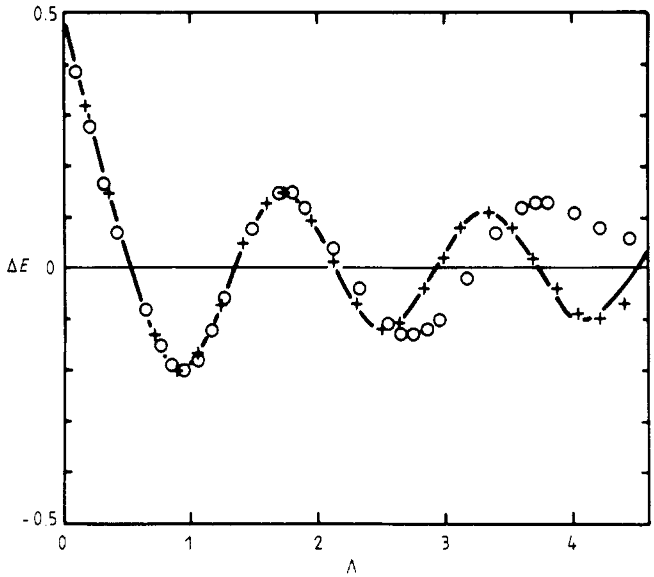

Figure 1. Numerical test of the scaling law for the energy levels. Deviations ΔE of the energy levels from their corresponding baselines $E_\nu = \nu$ for $\Delta = 1 (n_\Delta = 0)$. Full curve: $\Delta E = \frac{1}{2}\bar{\Delta}_0$ against Λ for the effective Hamiltonian. Circles and crosses: $\Delta E = \frac{1}{2}\Delta_0$ against the scaled coupling $\Lambda = \lambda(\nu + \frac{1}{2})^{1/2}$ for the two-level system for the levels $\nu = 4$ (circles) and $\nu = 12$ (crosses).

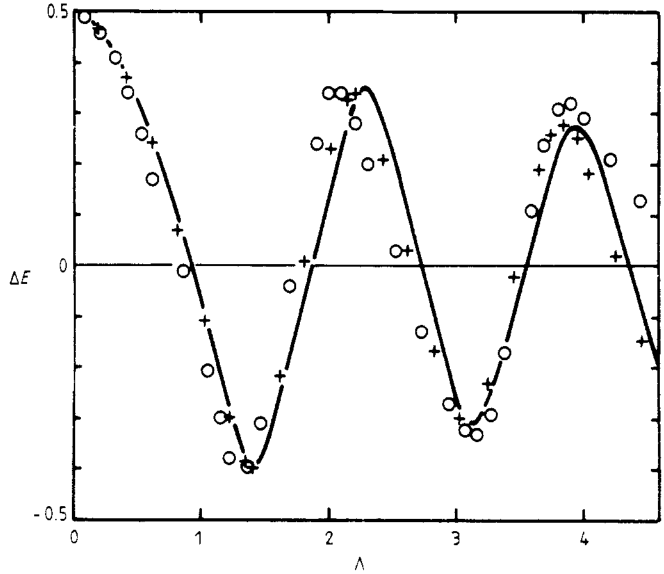

Figure 2. The same as figure 1 for $\Delta = 3$, $\nu = 5$ (circles) and $\nu = 17$ (crosses). Because here $n_\Delta = 1$, the scaled coupling is $\Lambda = \lambda(\nu - \frac{1}{2})^{1/2}$.

We see from (4.13) that the spectrum of $\bar{H}$ is completely determined by the knowledge of the function $\frac{1}{2}\bar{\Delta}_0(\Lambda, \Delta)$ which describes up to the sign the deviation of the levels from their corresponding baselines $E_\nu = \nu$.

Except for the trivial cases $\Delta = 0$ and $\Lambda = 0$ the deviation $\frac{1}{2}\bar{\Delta}_0$ is not known analytically. The perturbative treatment of Δ is very instructive because the resulting expression shows many characteristic properties of the general case. Here we assume $|\Delta| \ll 1$ and, hence, $n_\Delta = 0$ and $\Delta = \Delta_0$. We obtain

$$\tfrac{1}{2}\bar{\Delta}_0(\Lambda, \Delta) = \tfrac{1}{2}\Delta J_0(4\Lambda). \tag{4.16}$$

Explicit approximate solution for various regimes of Λ and Δ have been given by Autler and Townes (1955). Being interested in the overall behaviour of $\bar{\Delta}_0(\Lambda, \Delta)$ we retreat to numerical calculations. In the basis $\{|n\rangle\}$ the matrix representation of $\bar{H}$ is tridiagonal. For the determination of $\bar{\Delta}_0(\Lambda, \Delta)$ we only need to know a single eigenvalue. An isolated eigenvalue can be obtained very efficiently by use of the node theorem for the eigenvectors of tridiagonal matrices (for details see ch 2 of Lieb and Mattis (1966)). After a symmetric truncation at $n = \pm m_0$ we determine the eigenvalue next to $E = 0$ for which the effects of the truncation are minimal. Avoiding in this way the diagonalisation of the whole matrix we obtain $\bar{\Delta}_0(\Lambda, \Delta)$ with sufficient accuracy from rather low dimensional approximations (typically $m_0 \approx 30$).

The full curves of figures 1 and 2 show numerical results for $\Delta = 1$ and $\Delta = 3$. We arrive at the following properties of $\bar{\Delta}_0(\Lambda, \Delta)$ which we suppose to hold for arbitrary values of Δ: (i) $|\bar{\Delta}_0| \leq 1$ for all Λ and Δ; (ii) as a function of Λ the deviations show an infinite number of oscillations which die out for $\Lambda \to \infty$; (iii) with increasing values of $|\Delta|$ the maxima of $|\bar{\Delta}_0(\Lambda, \Delta)|$ increase accompanied by a decrease of the damping rate, and (iv) for large values of Λ the zeros of the deviations become equally spaced. As a consequence of (i) each level lies completely within a strip of half-width $\frac{1}{2}$ centred around the corresponding baseline.

5. Asymptotic scaling law for the energy levels and level distribution

The spectrum of $H(\lambda = 0, \Delta)$ coincides with that of $\bar{H}(\Lambda = 0, \Delta)$ for $n \geq 0$. We therefore may introduce for the Hamiltonian H (4.1) the level numbering ν well ordered with respect to the energy along the lines as was done for $\bar{H}$ in the last section. As a result, the spectrum of H may be written as

$$E(\nu, \lambda, \Delta) = \nu + \tfrac{1}{2}\Delta_0(\nu, \lambda, \Delta)(-1)^{\nu + n_\Delta} \qquad \nu \geq |n_\Delta| \tag{5.1}$$

$$\Delta_0(\nu = |n_\Delta|, \lambda = 0, \Delta) = \Delta_0 \tag{5.2}$$

where n_Δ and Δ_0 are defined by (4.10). Being interested in the large n limit which is equivalent to the large ν limit we henceforth consider only the levels with $\nu \geq |n_\Delta|$. This restriction of the parametrisation (5.1) and of the expressions (4.14) and (4.15) for $\nu(n)$ and $n(\nu)$ is due to the existence of a well defined ground state where $\nu = 0$.

According to numerical calculations (Reik *et al* 1982, Graham and Höhnerbach 1984a) the spectrum (5.1) has quite a complex structure. Nevertheless, it shows some characteristic features, for example as a function of λ and ν. When following an energy level from $\lambda = 0$ to $\lambda = \infty$ one can distinguish three regimes. In the first one the level oscillates around its corresponding baseline $E_\nu = \nu$. The number of these oscillations increases proportional to ν stretching to higher values of λ. The crossings with the baseline coincide with the exact solutions of Reik *et al* (1982). This regime is followed by an approach to the baseline which is finally reached for $\lambda=\infty$. For fixed values of λ and Δ we always enter the oscillatory regime when ν is increased to sufficiently large values. We therefore expect this regime to be related to the large $n(\nu)$ limit.

Some of the essential features of the levels are already present when Δ is treated as a perturbation. Assuming $|\Delta| \ll 1$ which implies $n_\Delta=0$ we obtain from (3.3) and (3.4)

$$\tfrac{1}{2}\Delta_0(\nu, \lambda, \Delta)=\tfrac{1}{2}\Delta\, e^{-2\lambda^2} L_\nu^{(0)}(4\lambda^2). \tag{5.3}$$

In the large ν limit we conclude quite generally from the comparison of (5.1) and (4.13) the following scaling law:

$$\tfrac{1}{2}\Delta_0(\nu, \lambda, \Delta)=\tfrac{1}{2}\bar{\Delta}_0(\Lambda(\lambda, \nu), \Delta) \tag{5.4}$$

where by the use of (3.6) and (4.14) the coupling of the effective Hamiltonian $\bar{H}_\nu$ is given by

$$\Lambda(\lambda, \nu)=\lambda[\nu+\tfrac{1}{2}-n_\Delta(-1)^{\nu+n_\Delta}]^{1/2}. \tag{5.5}$$

Hence, in the limit $\nu \to \infty$ the deviations from the baselines fall for given Δ onto the universal curve $\frac{1}{2}\bar{\Delta}_0(\Lambda, \Delta)$ when the coupling is scaled according to the rule (5.5). Numerical tests of this scaling law are presented in figure 1 for $\Delta=1$ and in figure 2 for $\Delta=3$. In the oscillatory regime of the levels the agreement with the asymptotic behaviour is remarkably good, even for low values of ν.

Let us now consider for given values of λ and Δ the nearest-neighbour spacings

$$\Delta E_\nu=E(\nu+1, \lambda, \Delta)-E(\nu, \lambda, \Delta). \tag{5.6}$$

In the large ν limit

$$\Delta E_\nu=1+\bar{\Delta}_0(\lambda\nu^{1/2}, \Delta)(-1)^{\nu+n_\Delta+1} \tag{5.7}$$

where we have set $\Lambda(\lambda, \nu+1) \approx \Lambda(\lambda, \nu) \approx \lambda\nu^{1/2}$. Hence, for $\nu \to \infty$ the level distribution is quite regular. For consecutive values of ν the spacings lie alternately on the two curves

$$\Delta_\pm E_\nu=1 \pm \bar{\Delta}_0(\lambda\nu^{1/2}, \Delta). \tag{5.8}$$

Thus, the scaling law for the energy levels implies a scaling law for the nearest-neighbour spacing distribution. Suppose we choose a specific value for Δ. Then a comparison of (5.1) and (5.4) with (5.8) shows that following an energy level ν by varying λ gives the same information as the level distribution ΔE_ν for fixed λ. As long as the condition $\nu \gg \lambda^2$ is fulfilled we measure in both cases $\bar{\Delta}_0(\Lambda, \Delta)$ as a function of Λ. Furthermore, it follows under the same conditions that we obtain no new information when we consider different energy levels or the level distribution for different values of λ.

We make some additional remarks on the level distribution. For given Δ and λ we always enter the large ν limit when $\nu \gg \lambda^2$ where the scaling law applies. Making use of observations (i)–(iv) on the behaviour of $\bar{\Delta}_0(\Lambda, \Delta)$ mentioned at the end of § 4 we arrive at the following properties. For $\nu \to \infty$ the functions $\Delta_\pm E_\nu$ show damped oscillations around the mean value $\Delta E=1$ with $0 \leq \Delta_\pm E_\nu \leq 2$ and $\lim_{\nu\to\infty} \Delta_\pm E_\nu=1$. For small values of Δ the oscillations die out rather quickly. On the other hand, when Δ is large the damping is small and the functions $\Delta_\pm E_\nu$ nearly approach the bounds $\Delta E=0$ and $\Delta E=2$ over many periods. In addition, it follows from (iv) that the period of the oscillations roughly increases proportional to $\nu^{1/2}$ for increasing values of ν.

We obtain a coherent picture of ΔE_ν for all values of ν when we take into account the behaviour of the levels in the strong coupling limit $\lambda^2 \gg \nu$ where they have settled down to the baselines $E_\nu = \nu$. First assume $\lambda^2 \gg 1$. Then the lowest levels fall into the regime of the strong coupling limit and $\Delta E_\nu \approx 1$. When ν is increased to values of the order of λ^2 then ΔE_ν corresponds to levels which are approaching the baselines. Here ΔE_ν is found to be quite irregular. For $\nu \gg \lambda^2$ we finally enter the large ν regime described above where ΔE_ν has split into the two functions $\Delta_\pm E_\nu$. In the opposite case $\lambda^2 \ll 1$ the distribution ΔE_ν directly begins with the irregular part. In any case, aside from a limited range, the distribution of the nearest-neighbour spacings is a quite regular function of ν.

These properties are in complete agreement with the numerical results obtained by Kuś (1985a) (see also Steeb *et al* 1985). Because these properties form the basis for the conclusion of Kuś (1985a) and Steeb *et al* (1985) that the spectrum of the two-level system indicates the non-existence of 'quantum chaos' we may say that our results provide an analytic support of this view. As was emphasised by Kuś (1985a) however, this conclusion relies on a tentative definition of 'quantum chaos'. Nevertheless, the rather complete understanding we have obtained of the characteristic properties of the level distribution renders the two-level system into a convenient model for the application of any criterion based on the sequence of eigenvalues indepedent of its eventual final form.

6. Interpretation of the effective Hamiltonian as a tight-binding model for the two-band Stark ladder

We add a few general remarks on the effective Hamiltonian (4.2). In the basis of the occupation number states $\bar{H}$ may be written as

$$\bar{H} = \sum_{-\infty}^{+\infty} \varepsilon(n)|n\rangle\langle n| + \Lambda \sum_{-\infty}^{+\infty} (|n\rangle\langle n+1| + \mathrm{cc}) \tag{6.1}$$

where

$$\varepsilon(n) = \varepsilon_0(n) + \varepsilon_1(n) \tag{6.2}$$

$$\varepsilon_0(n) = \tfrac{1}{2}\Delta(-1)^n \qquad \varepsilon_1(n) = n. \tag{6.3}$$

Written in the form (6.1) $\bar{H}$ has the obvious interpretation as a one-dimensional tight-binding model with local energies $\varepsilon(n)$ and the hopping matrix element Λ. The contribution of $\varepsilon_1(n)$ may be interpreted as the potential of a homogeneous electric field.

In the absence of this field we face a system with period 2. The corresponding energy bands $\bar{E}^0_s(k)$ and Bloch states $|k, s\rangle$ with $s = \pm 1$ are

$$\bar{E}^0_s(k) = s[\tfrac{1}{4}\Delta^2 + (2\Lambda\cos k)^2]^{1/2} \tag{6.4}$$

$$|k, s\rangle = \begin{cases} \pi^{-1/2} \sum_{-\infty}^{+\infty} \exp(\mathrm{i}k2n)|2n\rangle & \text{for } s = +1 \\ \pi^{-1/2} \sum_{-\infty}^{+\infty} \exp[\mathrm{i}k(2n+1)]|2n+1\rangle & \text{for } s = -1. \end{cases} \tag{6.5}$$

The application of a homogeneous field to a system with s bands has the well known effect that all states become localised, reflected by a discrete spectrum consisting of s Stark ladders (Avron and Zak 1977) (see also Avron 1982). In our case the two Stark ladders are those described in § 4.

With k from the first Brillouin zone $[0, \pi]$ the Bloch states $|k, s\rangle$ form a complete set of states and for an arbitrary state we have

$$|\psi\rangle = \sum_s \int_0^{\pi} \mathrm{d}k\, u_s(k)|k, s\rangle \tag{6.6}$$

with the boundary condition

$$u_s(\pi) = s u_s(0). \tag{6.7}$$

Introduce a two-component wavefunction

$$u(k) = \begin{pmatrix} u_+(k) \\ u_-(k) \end{pmatrix} \tag{6.8}$$

then the time-independent Schrödinger equation becomes

$$(\mathrm{i}\partial_k + 2\Lambda \cos k\sigma_x + \tfrac{1}{2}\Delta\sigma_z)u(k) = \bar{E}u(k) \tag{6.9}$$

supplemented by the boundary condition

$$u(\pi) = \sigma_z u(0). \tag{6.10}$$

Now interpret the momentum k as the time t, then the eigenvalue equation (6.9) takes the form of a time-dependent Schrödinger equation

$$\mathrm{i}\partial_t u(t) = h(t)u(t) \tag{6.11}$$

$$h(t) = -2\Lambda \cos t\sigma_x - \tfrac{1}{2}\Delta\sigma_z + \bar{E}. \tag{6.12}$$

Recalling that in the large n limit $\Lambda = \lambda n_0^{1/2}$ we see that (6.11) and (6.12) describe the dynamics of a two-level system in a strong oscillating field. Starting from the effective Hamiltonian $\bar{H}$ we have thus arrived at the time-dependent formulation of the large n limit, as was considered in detail by Autler and Townes (1955) and Shirley (1965). Note, however, that $\bar{H}$ refers to the Hamiltonian (4.1) which includes the shift λ^2.

7. Conclusion

The final version of the effective Hamiltonian in the large n limit (4.2) differs from the original problem written in the form (4.1) in two ways. First, by working in the extended Hilbert space we get rid of the irrelevant boundary effects which affect only the low lying levels. Second, the transition from the Bose operators a, a^+ to the commuting operators $\bar{T}$, $\bar{T}^+$ gives rise to an appreciable simplification which is reflected by a number of symmetry properties. The scaling law for the highly excited levels, together with its implications for the level distribution, is a direct consequence thereof. On the other hand, the transition to commuting operators introduces an operator ordering problem which is not without subtleties, as was pointed out at the end of § 3. We note that the 'C number properties' of $\bar{T}$ and $\bar{T}^+$ are intimately related to the fact that they are unitary and, hence, bounded operators. This indicates once more that the failure of the usual formulation of the large n limit results from an improper approximation of the unbounded Bose operators by bounded ones. We have avoided this problem by means of a unitary transformation.

The content of § 6 may be summarised as follows: the large n limit of a two-level system coupled to a boson mode, the dynamics of a spin in a strong oscillating field and the two-band Stark effect are equivalent formulations of one and the same problem.

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